Formative Conversation Starters: Math

GRADE 6

nwea Professional Learning

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Formative Conversation Starters

Student understanding is more about growing than it is about getting. As educators, we might speak of some students "getting it" and other students not. This conclusion, though, is not fair to students. A student who may not seem to "get it" does understand *something*, even if that *something* may not yet have grown into a robust web of thinking. To reach students' full knowledge, we can use guided conversations: Formative Conversation Starters.

Purpose: The purpose of Formative Conversation Starters is to help teachers reveal student understanding about key ideas in mathematics and to identify their students' ways of thinking.

Audience: The intended audience is teachers, who will use these questioning strategies with students.

Application: Teachers may wish to use these conversation starters in one-on-one conferences with students or in small groups.

Formative Conversation Starters approach student knowledge by presenting a single standards-based assessment item and leveraging the item to elicit conversation through clustered questioning. The goal of this activity is not to tell students what to think, but to help teachers better uncover how students are currently thinking about mathematical concepts. The conversations provide opportunities for students to communicate how they are thinking about mathematics.

Mathematical Ideas (BINSS – Big Ideas to Nurture Standards Sense-making)

As you read through these items, you will notice that we draw attention to a few specific mathematical ideas. These ideas correspond to important ways of thinking that all students should develop and continue to refine. They include:

- **Operations:** Students begin to develop meanings for operations in kindergarten (e.g., addition is putting together). As they progress, the numbers involved—and operational meanings—extend. Students should develop ways of thinking that enable them to connect operation meanings to everyday use of those operations. Operations should never be disconnected from meaning. Division of fractions, for example, is still a form of division and should connect to a meaning of division.

- **Place Value:** Knowledge of place value is essential, and students should develop ways of thinking about place value that enable them to see the relationships between places. For example, they can think of a value in one place as 10 times that same value in the place to the right (*or a bundle of 10*), and they can carry that thinking between any places in any direction. They should be able to use that understanding effortlessly to compose and decompose quantities and to connect place-value understanding to operations.

- **Comparisons:** Comparisons can be either additive or multiplicative, with context guiding which is most appropriate. A multiplicative comparison is relative, describing one quantity in terms of another (e.g., 6 meters is 3 times as large as 2 meters). Additive comparisons are absolute; the comparison is based on some other quantity (e.g., 6 meters is 4 meters more than 2 meters). Students should have ways of thinking that help them determine which comparison to use or how an existing comparison is additive or multiplicative.

- **Measurement:** Geometric measurement is ultimately understood as the result of a multiplicative comparison between common attributes of two measurable quantities, and the result describes how many copies of a are contained in b. Equivalently, measurement addresses a times-as-large comparison such as "a is n times as large as b." Students thinking about measurement should have a clear understanding of which attribute is being measured and the comparison of two objects with that attribute, where one object's attribute is measured in terms of the other.

- **Fractions:** A fraction is a single number. It is a number just as 1, or 100, or 37,549 are, and it has a location on the number line. Students should be able to think of a fraction as a number and treat it as such. The fraction a/b can be thought of as a copies of $1/b$, where $1/b$ is the length of a single part when the interval from 0 to 1 is partitioned into b parts. Two fractions are equivalent when they share a location on the number line.

- **Formulas:** Mastery of formulas (and procedures) is not the goal. Formulas are not the mathematics; they should be seen as shortcuts to help accomplish something with the mathematics. Students should have ways of thinking about the formulas that enable them to make sense of the quantities and to determine why quantities are connected with the indicated operations. Students should have mental and mathematical ways to reinvent useful formulas (e.g., $A=\pi r2$ means three-and-a-bit copies of the square with area $r2$).
- **Variables:** Students should have ways of thinking that enable them to distinguish between unknowns that vary and unknowns that represent some fixed value. For example, in the equation $6=3x+2$, x represents some unknown fixed value that makes the equation true. In $y=3x+2$, y and x vary with each other. In $y=mx+b$, y and x vary with each other, while m and b are typically nonvarying constants (parameters) within a problem.
- **Covarying Quantities:** Single quantities can vary, but students also need to consider situations where two quantities vary together. For example, in the equation $y=3x+2$, as the quantity x varies, the quantity y varies. It can be helpful to think about the relationship between covarying quantities in terms of how changes in x result in changes in y (e.g., as x increases by 1, y changes by . . .).
- **Proportional Relationships:** Proportional relationships require two covarying quantities. Those quantities must be measurable in some way, and the measures of those quantities scale in tandem. When one quantity changes by a scale factor, the other quantity also changes by the same scale factor. For example, doubling one quantity's value results in a doubling of the other quantity's value. Students should have ways of thinking that allow them to distinguish the two varying quantities in any proportional relationship and to explain how the quantities change by the same scale factor. It is important to note that proportional relationships are not synonymous with proportions. See this whitepaper for more information.
- **The Equal Sign:** The equal sign works in multiple ways in mathematics, even though the symbol does not change. Students should be able to think about the symbol as being relational (e.g., 2+4=5+1) and as being operational (e.g., the output of a computation), and should be able to determine which role the symbol is playing based on the situation. Keep an eye out for when students may put together equal signs. If this happens, prompting the student with questions about what the equals sign means may help. For example, suppose a student writes: 9 x 8 = 72 = 126 – 72 = 54 (*in this case, the student might be using the equal sign to say "next, I will…"*).

In the Formative Conversation Starters, you will notice that we call out mathematical ideas and the ways of thinking associated with them when it makes sense to do so. We also group discussion prompts to focus the conversations. Some groupings target core understandings that underpin the content. Other groupings elicit flexibility of thinking or extend beyond the assessment item being discussed. All of these methods are intended to provide opportunities for the teacher to listen to students and to reflect on how students might be thinking about mathematics in the standards-based assessment item.

The progressive question-and-answer strategy can be used to elicit evidence of students' ways of thinking about a topic or concept, with the purpose of guiding instruction.

How to Conduct a Formative Conversation

1. The questions for each item were developed to help teachers elicit information about students' ways of thinking about the content in the item and about mathematical ideas. These questions are suggestions, however, and not intended to be used as a script. The conversations teachers have will vary by student. While the questions for an item are laid out in a progression, teachers should vary the order to adapt to students' responses. Teachers should also keep in mind that students' responses may point to ways of thinking that are not addressed by the provided questions. In these cases, teachers should pursue those student understandings with their own line of questioning. There are several actions teachers can take to prepare for formative conversations:
    a. Become very familiar with the task and the questions ahead of time. This ensures that teachers can select the most appropriate next question based on how the students are responding.

   b. Provide students with tools to help them answer the questions. Depending on the task, these tools might include a manipulative, drawing paper, graph paper, or individual whiteboards and markers.
   c. Have a list of questions to help further probe what students are thinking. Some examples are:
      i. Can you tell me more about that?
      ii. You look like you're really thinking about this. What are you thinking?
      iii. Can you draw me a picture/write an equation?
      iv. How did you get that answer?
      v. Is there another way that you could find that answer?
   d. Make a plan to track what students say during the conversation: record the conversation, take notes, or have an observer take notes.
2. Teachers should ask questions without judgment. Student responses should not be labeled as right or wrong, and follow-up questions should be asked regardless of whether students give a correct response. Teachers should avoid commenting on students' responses other than to ask follow-up questions or to clarify what a student has said. Other students, however, should be encouraged to agree or disagree in a small group setting.
3. One of the most important parts of the formative conversation is what comes after the conversation: how will a teacher use the information about student thinking when planning instruction. Consider these suggestions for how to act on a formative conversation:
   a. Identify the different mathematical ideas addressed in the formative conversation. Where did students make connections between the ideas? Where do the connections need to be strengthened?
   b. Identify what students already understand in order to build instruction on that understanding.
   c. Identify areas where students can deepen what they already understand.
   d. Identify ways that students are comfortable with expressing mathematical ideas, and plan how to expand their capabilities. How were students most comfortable expressing or explaining what they understand? Were they more inclined to create a graphical representation of their thinking? Did they prefer to explain verbally? Did they use equations, or did they prefer to use graphs?
4. Some Conversation Starters include extension questions. These questions are provided as a way to elicit thinking beyond the grade-level of the BINSS.

6.1 Proportional Relationships

This activity focuses on student thinking about ratios.

Proportional Relationships: Proportional relationships require two covarying quantities. Those quantities must be measurable in some way, and the measures of those quantities scale in tandem. When one quantity changes by a scale factor, the other quantity also changes by the same scale factor. For example, doubling one quantity's value results in a doubling of the other quantity's value. Students should have ways of thinking that allow them to distinguish the two varying quantities in any proportional relationship and to explain how the quantities change by the same scale factor. It is important to note that proportional relationships are not synonymous with proportions. See **this whitepaper** for more information.

ITEM ALIGNMENT

CCSS: 6.RP.A.3

This item focuses on ratios. However, it also provides an opportunity to talk about unit rates, reasoning about ratios and rates, and interpreting ratios.

THE CONVERSATION STARTER

Use the information to complete the task.

Ramiro made a drink by mixing 4 tablespoons (Tbsp) of chocolate syrup with 12 fluid ounces (fl oz) of milk.

Choose all the mixtures that have the same ratio of chocolate syrup to milk as Ramiro's drink.

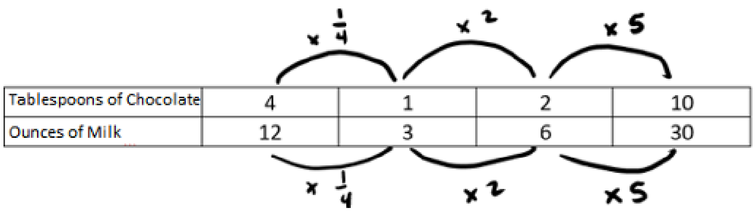
- ☐ A. 2 Tbsp of chocolate syrup mixed with 6 fl oz of milk
- ☐ B. 5 Tbsp of chocolate syrup mixed with 13 fl oz of milk
- ☐ C. 6 Tbsp of chocolate syrup mixed with 10 fl oz of milk
- ☐ D. 8 Tbsp of chocolate syrup mixed with 16 fl oz of milk
- ☐ E. 10 Tbsp of chocolate syrup mixed with 30 fl oz of milk

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Content: Ratio (Meaning, Equivalence)

What must happen to keep two measures in the same ratio?

What does “same ratio” mean in the sentence, “Choose all the mixtures that have the same ratio of chocolate syrup to milk as Ramiro’s drink”?



Tablespoons of Chocolate	4	1	2	10
Ounces of Milk	12	3	6	30

B. Content: Ratio (Representation)

Can you represent the relationship between tablespoons of chocolate syrup and ounces of milk using a diagram? How would you start?

C. Content: Unit Rate

What amount of milk would we need for 1 tablespoon of chocolate syrup? How did you figure it out?

- How does knowing the amount of milk needed for 1 tablespoon of chocolate syrup help us figure out different amounts?
- What amount of chocolate syrup would we need for 1 ounce of milk?
- How much chocolate syrup would we need if we had 1,200 ounces of milk? Explain.

D. Content: Ratio (Equivalence)

Ramiro’s sister wants her drink to have the same sweetness as his drink. Which ratios of syrup to milk could she use?

If Ramiro’s sister makes a drink with 4 tablespoons of chocolate syrup mixed with 16 ounces of milk, will her drink be sweeter than Ramiro’s drink?

If Ramiro’s sister makes a drink with 6 tablespoons of chocolate syrup mixed with 14 ounces of milk, will the drink be sweeter than Ramiro’s drink?

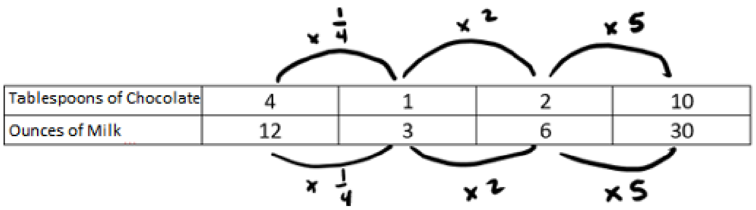
CONVERSATION PATHS (ANNOTATED)*

A. Content: Ratio (Meaning, Equivalence)

What must happen to keep two measures in the same ratio?

> ***If one measure is changed by a factor of k, then the other also has to change by the same factor of k.***

What does "same ratio" mean in the sentence, "Choose all the mixtures that have the same ratio of chocolate syrup to milk as Ramiro's drink"?



Tablespoons of Chocolate	4	1	2	10
Ounces of Milk	12	3	6	30

> ***Students might answer this question in one or more of the following ways (based on the categories for thinking about ratios):***
>
> - ***Scaling: If the number of tablespoons is cut in half, the number of fluid ounces is cut in half.***
> - ***The values of ratios are equal: $4/12 = 2/6 = 10/30$***
> - ***Constant multiple: The number of fluid ounces (12) is 3 times the number of tablespoons (4). Or, the number of fluid ounces (6) is 3 times the number of tablespoons (2).***

B. Content: Ratio (Representation)

Can you represent the relationship between tablespoons of chocolate syrup and ounces of milk using a diagram? How would you start?

> ***Student diagrams will vary. A common tool that is used to make sense of ratios is a ratio table, as shown below. Ratio tables illustrate the "constant multiple" way of thinking.***

C. Content: Unit Rate

What amount of milk would we need for 1 tablespoon of chocolate syrup? How did you figure it out?

> ***Some students may think about scaling (cut both quantities by a factor of $1/4$), or a constant ratio ($4/12 = 1/3$ so 1 tablespoon corresponds to 3 fluid ounces), or a constant multiple (4 x 3 = 12 and 1 x 3 = 3).***

- How does knowing the amount of milk needed for 1 tablespoon of chocolate syrup help us figure out different amounts?

  When the unit rate is known, scaling is simpler.

- What amount of chocolate syrup would we need for 1 ounce of milk?

  Some students may think about scaling by $1/12$, resulting in $4*(1/12) = 1/3$ tablespoon. Or, some might use a constant ratio, thinking $4/12 = 1/3 = (1/3)/1$ so $1/3$ tablespoon of chocolate will correspond to 1 ounce of milk. Or, some might use a constant multiple, observing that the number of ounces of milk is 3 times the number of tablespoons of chocolate, so given 1 ounce of milk, $1/3$ of 1, or $1/3$ tablespoons of chocolate would be needed.

- How much chocolate syrup would we need if we had 1,200 ounces of milk? Explain.

  400 tablespoons. If we think about scaling, 1,200 ounces of milk is 100 times as large as 12 ounces of milk. The amount of chocolate must also be 100 times as large as 4 tablespoons of milk, so 400 tablespoons are needed. If we think about a constant multiple, 12 is 3 times 4 and 1,200 is 3 times 400, so 400 tablespoons.

D. Content: Ratio (Equivalence)

Ramiro's sister wants her drink to have the same sweetness as his drink. Which ratios of syrup to milk could she use?

> ***She could use any ratio with a value equal to $\frac{4}{12}$, such as $\frac{1}{3}$, $\frac{2}{6}$, or $\frac{3}{9}$.***

If Ramiro's sister makes a drink with 4 tablespoons of chocolate syrup mixed with 16 ounces of milk, will her drink be sweeter than Ramiro's drink?

> ***No. She will have added the same amount of syrup to a greater volume of milk, so her drink will be less sweet. When the value of the ratio of syrup to milk, which is multiplicative comparison, is the same for two drinks, then the two drinks will be of equal sweetness.***

If Ramiro's sister makes a drink with 6 tablespoons of chocolate syrup mixed with 14 ounces of milk, will the drink be sweeter than Ramiro's drink?

> ***Yes. We can think of the value of the given ratio, $\frac{4}{12}$ or $\frac{1}{3}$, as a measure of sweetness. A higher value would represent a sweeter mixture, while a drink with a lower value would be less sweet. From there, one might compare the value (or the location on the number line) of $\frac{6}{14}$ to $\frac{1}{3}$. It's possible to do this in a few different ways. If we scale the measures of the given ratio $\frac{1}{3}$ by 6, we can compare $\frac{6}{14}$ to $\frac{6}{18}$. We could also scale the measures in each ratio to find a common denominator, allowing a comparison of $\frac{18}{42}$ to $\frac{14}{42}$. Either way, the measure of sweetness is higher in Ramiro's sister's drink. Students who think that 6 to 14 is an equal ratio to 4 to 12 might only be thinking about the fact that 6 is 2 more than 4 and 14 is 2 more than 12.***

6.1 Shareables*

Use the information to complete the task.

Ramiro made a drink by mixing 4 tablespoons (Tbsp) of chocolate syrup with 12 fluid ounces (fl oz) of milk.

Choose all the mixtures that have the same ratio of chocolate syrup to milk as Ramiro's drink.

- [ ] A. 2 Tbsp of chocolate syrup mixed with 6 fl oz of milk
- [ ] B. 5 Tbsp of chocolate syrup mixed with 13 fl oz of milk
- [ ] C. 6 Tbsp of chocolate syrup mixed with 10 fl oz of milk
- [ ] D. 8 Tbsp of chocolate syrup mixed with 16 fl oz of milk
- [ ] E. 10 Tbsp of chocolate syrup mixed with 30 fl oz of milk

A.

Tablespoons of Chocolate	4	1	2	10
Ounces of Milk	12	3	6	30

6.2 Proportional Relationships and Comparisons

This activity focuses on student thinking about percent.

Proportional Relationships: Proportional relationships require two covarying quantities. Those quantities must be measurable in some way, and the measures of those quantities scale in tandem. When one quantity changes by a scale factor, the other quantity also changes by the same scale factor. For example, doubling one quantity's value results in a doubling of the other quantity's value. Students should have ways of thinking that allow them to distinguish the two varying quantities in any proportional relationship and to explain how the quantities change by the same scale factor. It is important to note that proportional relationships are not synonymous with proportions. See **this whitepaper** for more information.

Comparisons: Comparisons can be either additive or multiplicative, with context guiding which is most appropriate. A multiplicative comparison is relative, describing one quantity in terms of another (e.g., 6 meters is 3 times as large as 2 meters). Additive comparisons are absolute; the comparison is based on some other quantity (e.g., 6 meters is 4 meters more than 2 meters). Students should have ways of thinking that help them determine which comparison to use or how an existing comparison is additive or multiplicative.

ITEM ALIGNMENT

CCSS: 6.RP.A.3c

This item focuses on percent. However, it also provides an opportunity to talk about ratios and strategies for approaching a problem.

THE CONVERSATION STARTER

Use the information to answer the question.

Holly's recipe for birdseed is 80% sunflower chips. Holly has 16 ounces of sunflower chips.

How many ounces of birdseed can Holly make using her recipe? Enter the answer in the box.

[ ] ounces

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation & Strategy

What is this problem asking?

- What is the relationship between the birdseed and sunflower chips?
- The problem says that "80% of Holly's recipe for birdseed is sunflower chips." What does this mean? What is the other 20% of the recipe?
- What information do you need to find out?
- What are some ways to figure this out?
  - Could you use an equation?
  - How about a drawing, a chart, or a table?

B. Content: Percent

What is a percent?

- If 25% of a number is 10, what is the number?
- In what different ways could you find 25% of 200?
- If I tell you what 10% of a number is, can you tell me how to find 2%? 18%?

C. Content: Ratio (Meaning)

What might it mean if I say that the ratio of dogs to cats is 4:3?

- What does it mean for two ratios to be equivalent?
- Can you represent the birdseed situation using ratios?
- How can thinking about the relationship between the birdseed and sunflower chips as a ratio help solve the problem?

CONVERSATION PATHS (ANNOTATED)*

A. Problem Solving: Orientation & Strategy

What is this problem asking?

A bag of birdseed is typically made up of different kinds of seeds/chips. Holly has sunflower chips and other parts, so we need to figure out the number of ounces of the other parts to use and then the total ounces of birdseed she can make.

- What is the relationship between the birdseed and sunflower chips?

  The birdseed is the whole and the 16 ounces of sunflower chips are part of the whole. The amount of the recipe that is sunflower chips and the amount of the recipe that is another type of seed is a ratio of 80 to 20.

- The problem says that "80% of Holly's recipe for birdseed is sunflower chips." What does this mean? What is the other 20% of the recipe?

  In this case, 80 parts of the birdseed's 100 equal parts are sunflower chips. The other 20% is, presumably, some other kind of seed. The purpose of the question about the other 20% is to see what students understand about what the whole in this problem is.

- What information do you need to find out?

  The number of ounces of the other 20% of the recipe.

- What are some ways to figure this out?

  - Could you use an equation?

    One possible equation: 80% of x = 16 or ($\frac{4}{5}$)x = 16.

  - How about a drawing, a chart, or a table?

    For example:

16 oz	?
80%	20%
100%	

B. Content: Percent

What is a percent?

A percent is a multiplicative comparison of one quantity to another that is interpreted in terms of "per 100." We view the compared value to the whole as if the whole is scaled to a value of 100.

- If 25% of a number is 10, what is the number?

  Since 25% is ¼ of 100%, we know that 10 is ¼ of the whole. Therefore, 25% of 40 is 10.

- In what different ways could you find 25% of 200?

  We could find 10% of 200 (20) and 5% of 200 (half of 10%, so 10). Since 25% is 2 copies of 10% plus 5%, 25% of 200 is 20 + 20 + 10 = 50. Another option is to recognize that 25% represents ¼ of 100%. Therefore, 25% of 200 is ¼ of 200 (cut 200 into 4 equal parts), which is 50.

- If I tell you what 10% of a number is, can you tell me how to find 2%? 18%?

  We could determine 1% ($\frac{1}{10}$ of 10%) and then think about 2 copies of this 1%. For 18%, we could double the 10% and then subtract 2 copies of 1%.

C. Content: Ratio (Meaning)

What might it mean if I say that the ratio of dogs to cats is 4:3?

> ***It might mean that there are 4 dogs and 3 cats. It might also mean that there are 4 dogs for every 3 cats, which means there could be some other numbers of dogs and cats, such that both 4 and 3 are scaled by the same factor.***

- What does it mean for two ratios to be equivalent?

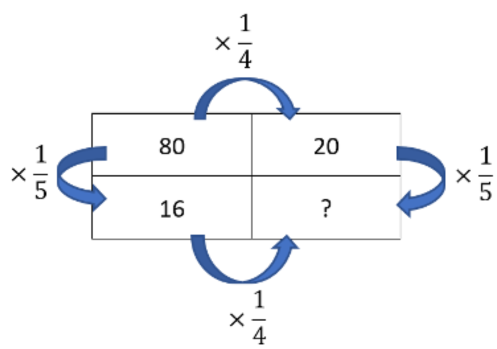
  If we can multiply the values in one ratio by the same number and the result is the other ratio, the two ratios are equal. For example, the ratio 4:3 is equivalent to 40:30 because we can multiple each value in 4:3 by 10 and the result is 40:30. Or, two ratios are equivalent if they have the same values. For example, $\frac{4}{3} = \frac{40}{30}$.

- Can you represent the birdseed situation using ratios?

  Do students recognize the 80:20 relationship between sunflower chips and other birdseed ingredients and connect it to the given 16 ounces? One ratio is 80:20 and the other is 16:_____.

- How can thinking about the relationship between the birdseed and sunflower chips as a ratio help solve the problem?

  If the ratio of sunflower chips to other chips is 80:20, then the ratio of the ounces of sunflower chips to other chips must be equivalent to that. One approach could be to recognize that 16 is $\frac{16}{80}$ (or $\frac{1}{5}$) of 80. As such, the number of ounces of the other chips must be $\frac{16}{80}$ (or $\frac{1}{5}$) of 20. Another approach would be to recognize that 80 is 4 times as large as 20, so the ounces of sunflower chips must be 4 times as large as the number of ounces of the other chips. Since 16 is 4 times as large as 4, there must be 4 ounces of other chips. Both ways of thinking are illustrated in the table below:



6.2 Shareables*

Use the information to answer the question.

Holly's recipe for birdseed is 80% sunflower chips. Holly has 16 ounces of sunflower chips.

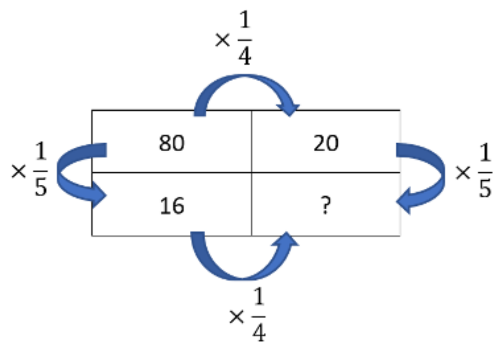
How many ounces of birdseed can Holly make using her recipe? Enter the answer in the box.

[ ] ounces

A.

16 oz	?
80%	20%
100%	

C.



6.3 Fractions and Operations

This activity focuses on student thinking about relating fractions.

Fractions: A fraction is a single number. It is a number just as 1, or 100, or 37,549 are, and it has a location on the number line. Students should be able to think of a fraction as a number and treat it as such. The fraction a/b can be thought of as a copies of $1/b$, where $1/b$ is the length of a single part when the interval from 0 to 1 is partitioned into b parts. Two fractions are equivalent when they share a location on the number line.

Operations: Students begin to develop meanings for operations in kindergarten (e.g., addition is putting together). As they progress, the numbers involved—and operational meanings—extend. Students should develop ways of thinking that enable them to connect operation meanings to everyday use of those operations. Operations should never be disconnected from meaning. Division of fractions, for example, is still a form of division and should connect to a meaning of division.

ITEM ALIGNMENT

CCSS: 6.NS.A.1

This item focuses on relating fractions. However, it also provides an opportunity to talk about meanings for division, scaling, and problem solving.

THE CONVERSATION STARTER

Use the information to answer the question.

Nami poured $\frac{2}{3}$ cup of rice into an empty jar. The rice filled $\frac{3}{5}$ of the jar.

How many cups of rice will fill an empty jar of the same size? Select and move numbers into the boxes. If there is no whole number, place 0 in the first box.

☐ $\frac{☐}{☐}$ cups

0 1 2 3 4 5 6 7 8 9 10

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation

What is this problem asking?

- What units are associated with the quantities in this problem? What unit is the whole for $\frac{2}{3}$? What unit is the whole for $\frac{3}{5}$?
- Can we say $\frac{2}{3}$ is equal to $\frac{3}{5}$?

B. Problem Solving: Strategy

How do you first want to think about solving this problem?

- What operation or operations could help you solve this problem? How would they help?

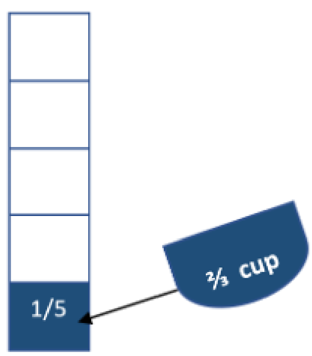
C. Content: Fractions (Division)

If I were to find a divided by b, is the answer always smaller than a?

- Can you explain what $1 \div \frac{1}{3}$ means?
- Can you explain what $1 \div \frac{2}{3}$ means?
- Suppose Jar A can fill $\frac{1}{3}$ of Jar B. How many times can a full Jar B fill Jar A?
- Suppose Jar A can fill $\frac{2}{3}$ of Jar B. How many times can a full Jar B fill Jar A?

D. Content: Fractions (Scaling)

What if $\frac{2}{3}$ cup of rice fills $\frac{1}{5}$ of the jar? How many cups of rice will fill an empty jar of the same size? Can you use the diagram to explain how you might go about solving this problem?



- From here, how does the answer change if $\frac{2}{3}$ cup fills $\frac{3}{5}$ of the jar?

CONVERSATION PATHS (ANNOTATED)*

A. Problem Solving: Orientation

What is this problem asking?

> ***The problem is asking how many cups of rice it will take to fill a jar when $\frac{2}{3}$ cup of rice fills $\frac{3}{5}$ of a jar.***

- What units are associated with the quantities in this problem? What unit is the whole for $\frac{2}{3}$? What unit is the whole for $\frac{3}{5}$?

  While the $\frac{2}{3}$ is measured in cups, the $\frac{3}{5}$ is measured in "jars." It is important to see if students understand this distinction or if they read the problem and think that the $\frac{3}{5}$ is also measured in cups.

- Can we say $\frac{2}{3}$ is equal to $\frac{3}{5}$?

  If we specified that $\frac{2}{3}$ of a cup fills $\frac{3}{5}$ of a jar, we could say that they are equal in that respect. However, they do not represent the same whole, so we could not write $\frac{2}{3} = \frac{3}{5}$.

B. Problem Solving: Strategy

How do you first want to think about solving this problem?

> ***This question is intended to see how students are thinking and reasoning about the problem upon their first read of it.***

What operation or operations could help you solve this problem? How would they help?

> ***Division could make sense. For example, consider easier numbers (a problem-solving strategy): If 12 cups filled 4 jars, then 12 ÷ 4 could tell us how many cups are in one full jar. Similarly, if 12 cups filled $\frac{1}{4}$ of a jar, 12 ÷ $\frac{1}{4}$ could still tell us how many cups are in one full jar. That is, division can tell us the whole when we know the fractional part, such as to answer "12 is one-fourth of the whole, what is the whole?" Multiplication could also make sense, specifically in terms of scaling. For example, scaling both by $\frac{5}{3}$ will yield the number of cups to fill 1 jar. A longer chain of reasoning might be that if $\frac{2}{3}$ cup fills $\frac{3}{5}$ of a jar, scaling by a factor of $\frac{1}{2}$ leads to $\frac{1}{3}$ cup filling $\frac{3}{10}$ of a jar. Then, scaling by a factor of $\frac{1}{3}$ leads to $\frac{1}{9}$ cup filling $\frac{1}{10}$ of a jar. Then, scaling by a factor of 10 leads to 1 jar filled with $\frac{10}{9}$ cups or 1 and $\frac{1}{9}$ cups.***

C. Content: Fractions (Division)

If I were to find a divided by b, is the answer always smaller than a?

> ***If we are thinking about division as a comparison, then the quotient is not always smaller than a. If we are thinking about division only as cutting into smaller groups (with b as a whole number) it may not be as clear.***

- Can you explain what 1 ÷ $\frac{1}{3}$ means?

  One possibility is that it tells us how many copies of $\frac{1}{3}$ are needed to make 1. One could think about this in terms of measuring cups: How many $\frac{1}{3}$ cup measuring cups (scoops) are needed to create 1 full cup of flour? Three cups are needed. One might also think about how this expression can be used to find the answer to a question like "1 is $\frac{1}{3}$ of what I need, how much do I need?"

- Can you explain what 1 ÷ $\frac{2}{3}$ means?

  One possibility is that it tells us how many copies of $\frac{2}{3}$ make 1. Since we are dividing by something twice as large as in the previous question, the answer should be half the size. That is, it takes $\frac{3}{2}$, or $1\frac{1}{2}$, copies of $\frac{2}{3}$ to make 1. One might also think about how this expression can be used to find the answer to a question like "1 is $\frac{2}{3}$ of what I need, how much do I need?"

- Suppose Jar A can fill $\frac{1}{3}$ of Jar B. How many times can a full Jar B fill Jar A?

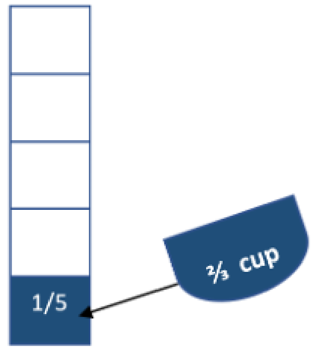
  3, based on the above.

- Suppose Jar A can fill $\frac{2}{3}$ of Jar B. How many times can a full Jar B fill Jar A?

  $\frac{3}{(2)}$, or $1\frac{1}{(2)}$, based on the above.

D. Content: Fractions (Scaling)

What if $\frac{2}{3}$ cup of rice fills $\frac{1}{5}$ of the jar? How many cups of rice will fill an empty jar of the same size? Can you use the diagram to explain how you might go about solving this problem?



One can scale both quantities by a factor of 5 to determine that 1 jar is filled by 5 times $\frac{2}{3}$ cup or $\frac{10}{3}$ cup or 3 $\frac{1}{3}$ cups.

- From here, how does the answer change if $\frac{2}{3}$ cup fills $\frac{3}{5}$ of the jar?

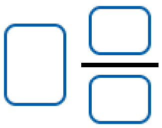
  We would need $\frac{1}{3}$ of the previous answer, or $\frac{1}{3}$ of $\frac{10}{3}$, or $\frac{(10)}{9} = 1\frac{1}{9}$.

6.3 Shareables*

Use the information to answer the question.

Nami poured $\frac{2}{3}$ cup of rice into an empty jar. The rice filled $\frac{3}{5}$ of the jar.

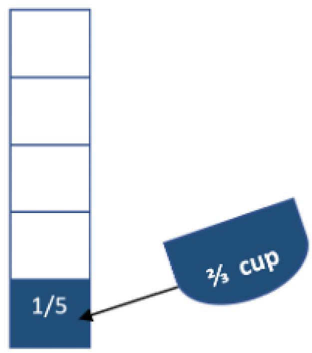
How many cups of rice will fill an empty jar of the same size? Select and move numbers into the boxes. If there is no whole number, place 0 in the first box.



cups

0 1 2 3 4 5 6 7 8 9 10

D.



6.4 Variables

This activity focuses on student thinking about using variables and expressions to solve problems.

Variables: Students should have ways of thinking that enable them to distinguish between unknowns that vary and unknowns that represent some fixed value. For example, in the equation $6 = 3x + 2$, x represents some unknown fixed value that makes the equation true. In $y = 3x + 2$, y and x vary with each other. In $y = mx + b$, y and x vary with each other, while m and b are typically nonvarying constants (parameters) within a problem.

ITEM ALIGNMENT

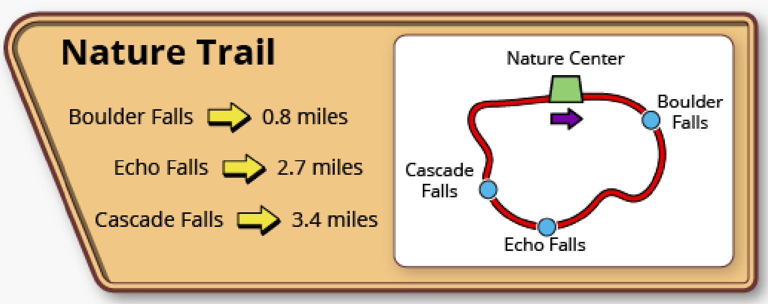
CCSS: 6.EE.B.6

This item focuses on using variables and expressions to solve problems. However, it also provides an opportunity to talk about variables and equations and to connect meanings to operations and quantities.

THE CONVERSATION STARTER

This question has two parts. Use the information to answer Part A and Part B.

Mona is hiking on a trail, starting at the Nature Center. The trail is 5.0 miles long. The sign shows the distances from the Nature Center to the waterfalls along the trail.



Mona stops at each waterfall along the trail, starting with Boulder Falls.

Part A

How many miles does Mona have left to hike on the trail after each stop? Enter the answers in the boxes to complete the table.

Waterfall	Number of Miles Left on the Trail
Boulder Falls	
Echo Falls	
Cascade Falls	

Part B

Which expression represents the number of miles Mona has left to hike on the trail after hiking m miles?

- A. $5 - m$
- B. $m - 5$

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation

How can you make sense of this problem situation?

- Can you think of another way to approach this problem?
- What is this problem asking you to do?

B. Content: Operations (Subtraction)

How many miles is it from Boulder Falls to Cascade Falls?

- How did you figure that out?
- Why did you use subtraction?

How many miles is it from Boulder Falls to the end of the loop?

- How did you figure that out?

What could 5 – 2.7 represent?

- What could 0.8 + 2.7 represent?

C. Content: Expressions (Comparison)

Look at these two statements. How are they the same? How are they different?

$$5 + y \qquad 5 + y = 2$$

D. Content: Expressions (Representation)

If g represents the weight, in ounces, of a package I am sending, what could the 10 possibly mean in each of the following expressions? What would the whole expression mean?

In the original problem, Part B asks you to identify the expression that can be used to find the number of miles left on the trail after hiking m miles. How might this expression help you?

- What if the distance from the Nature Center to Boulder Falls was 1.2 miles? Would that change the expression that you would use to find the number of miles left on the trail after hiking m miles?
- What if the whole trail was 6 miles long? What would the expression be to find the number of miles left on the trail after hiking m miles?

Let's look at another problem. What expression could we write to complete the table?

Hours	Depth in feet
0	4
1	5
5	9
h	

- What assumptions did you make to form that expression?

CONVERSATION PATHS (ANNOTATED)*

A. Problem Solving: Orientation

How can you make sense of this problem situation?

> ***The questions in this section are to help uncover students' understanding of the problem and what they might do to solve it. For example, students might begin by writing the mileage on the diagram and start to calculate any missing mileage.***

- Can you think of another way to approach this problem?

  Another possibility is that a student may move directly into the questions and indicate that the mileage to each falls can be subtracted from 5.

- What is this problem asking you to do?

  Possible answers may include being asked to find the mileage left to hike at each falls and writing an expression that models this.

B. Content: Operations (Subtraction)

How many miles is it from Boulder Falls to Cascade Falls?

> ***2.6 miles***

- How did you figure that out?

  Students may recognize that the focus is on ignoring the distance from the Nature Center to Boulder Falls, so 3.4 miles – 0.8 miles = 2.6 miles.

- Why did you use subtraction?

  Students may think that they are "removing" the distance from the Nature Center to Boulder Falls.

How many miles is it from Boulder Falls to the end of the loop?

> ***4.2 miles***

- How did you figure that out?

  5.0 – 0.8 = 4.2

What could 5 – 2.7 represent?

> ***The distance from Echo Falls to the end of the loop.***

- What could 0.8 + 2.7 represent?

  Adding the numbers in this situation doesn't give us any useful information, which is an important understanding. Just because we can add two numbers together doesn't mean doing so is helpful.

C. Content: Expressions (Comparison)

Look at these two statements. How are they the same? How are they different?

$$5 + y \qquad 5 + y = 2$$

> ***The first statement is called an expression, where the variable y is unknown. We do not know what y might represent, so the expression could take on any value. That value is 5 more than the value of y. The second statement is an equation because it has an equal sign and can be solved. We might refer to y as a variable here, but it does not really vary, as there is only one value of y that will make the equation true.***

D. Content: Expressions (Representation)

If g represents the weight, in ounces, of a package I am sending, what could the 10 possibly mean in each of the following expressions? What would the whole expression mean?

$$10g$$

$$g \div 10$$

$$10 + g$$

$$g - 10$$

Student responses will vary. There may be 10 packages and 10g could mean "the weight of 10 packages," or if the delivery service charges $0.10 per ounce, 10g could mean "the total cost of sending the package in cents." For the second expression, g divided by 10, the 10 could represent another package of 10 ounces and we are multiplicatively comparing the two packages. It might also represent breaking up the contents of our package into 10 equally weighted smaller packages. For 10 + g, it could mean the item plus 10 more ounces for a total weight of what is being sent. Here, the 10 must represent ounces. For g – 10, it could mean reducing the weight of the package by 10 ounces. Here, the 10 must also represent ounces.

In the original problem, Part B asks you to identify the expression that can be used to find the number of miles left on the trail after hiking m miles. How might this expression help you?

The purpose of this question is to listen for how students are thinking about the value and whether they can abstract a solution to a problem into an expression with a variable. Knowing that 5 – m can be used to find the total number of miles left in the trail can be used to solve the first three questions in Part A, as m can take on different values.

- What if the distance from the Nature Center to Boulder Falls was 1.2 miles? Would that change the expression that you would use to find the number of miles left on the trail after hiking m miles?

  The answer is no. But the purpose of the question is to see if students understand that the expression 5 – m can be applied to solve the problem regardless of the distance between the different waterfalls and the end of the trail.

- What if the whole trail was 6 miles long? What would the expression be to find the number of miles left on the trail after hiking m miles?

  The purpose of this question is to see if students understand what 5 – m means and if they can apply it to a trail that is a different number of miles.

Let's look at another problem. What expression could we write to complete the table?

Hours	Depth in feet
0	4
1	5
5	9
h	

h + 4

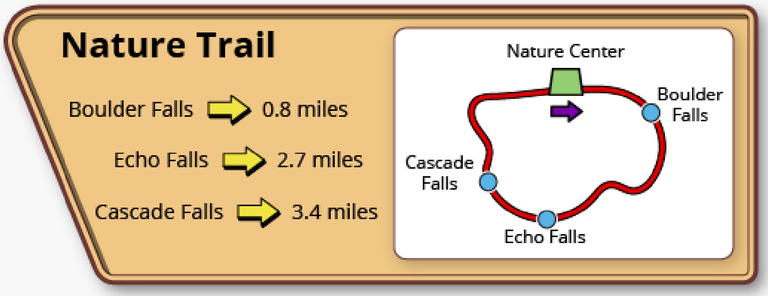
- What assumptions did you make to form that expression?

  Given the data we have, it looks like the depth is always four more than the number of hours. h + 4 might not be true for the data we don't have in the table. For example, if 4 hours corresponded to a depth of 7.5 feet, then h + 4 would not work.

6.4 Shareables*

This question has two parts. Use the information to answer Part A and Part B.

Mona is hiking on a trail, starting at the Nature Center. The trail is 5.0 miles long. The sign shows the distances from the Nature Center to the waterfalls along the trail.



Mona stops at each waterfall along the trail, starting with Boulder Falls.

Part A

How many miles does Mona have left to hike on the trail after each stop? Enter the answers in the boxes to complete the table.

Waterfall	Number of Miles Left on the Trail
Boulder Falls	
Echo Falls	
Cascade Falls	

Part B

Which expression represents the number of miles Mona has left to hike on the trail after hiking m miles?

- A. $5 - m$
- B. $m - 5$

C.

$$5 + y \quad 5 + y = 2$$

D.

$$10g$$
$$g \div 10$$
$$10 + g$$
$$g - 1$$

Hours	Depth in feet
0	4
1	5
5	9
h	

6.5 Variables

This activity focuses on student thinking about representing quantities using variables.

Variables: Students should have ways of thinking that enable them to distinguish between unknowns that vary and unknowns that represent some fixed value. For example, in the equation $6 = 3x + 2$, x represents some unknown fixed value that makes the equation true. In $y = 3x + 2$, y and x vary with each other. In $y = mx + b$, y and x vary with each other, while m and b are typically nonvarying constants (parameters) within a problem.

ITEM ALIGNMENT

CCSS: 6.EE.C.9

This item focuses on using variables to represent quantities in a real-world problem. However, it also provides an opportunity to talk about analyzing quantitative relationships, and relationships where measures scale in tandem.

THE CONVERSATION STARTER

Use the information to complete the task.

The table shows the number of cups of raspberries, r, needed to make p pies using a baker's recipe.

Number of Pies (p)	1	8	12	16	20
Number of Cups of Raspberries (r)	3	24	36	48	60

Choose "True" or "False" for each statement about the baker's recipe.

Statement	True or False?	
6 cups of raspberries are needed for 2 pies.	True	False
24 pies are made using 8 cups of raspberries.	True	False
The equation $p = 3r$ represents the relationship between r and p.	True	False
The point $(2, 6)$ would be on the graph that shows the relationship between the number of pies (shown on the horizontal axis) and the number of cups of raspberries (shown on the vertical axis).	True	False

CONVERSATION PATHS (QUESTIONS ONLY)*

A. Problem Solving: Orientation

How do you want to make sense of this problem?

- How many quantities are there in the question? What are they?
- Do the quantities seem to vary with each other? If so how?

B. Problem Solving: Strategy

What patterns do you see in the measurement data? What is the relationship between cups of raspberries and pies?

- What do each of those patterns mean?
- How is knowing this helpful?
- Could we use this relationship between the number of pies and the number of cups of raspberries to find any pair of measures for the pies and raspberries?
- Explain how the measures for 8 pies and 20 pies are related.

C. Content: Proportional Relationship

How many pies can I make with 300 cups of raspberries?

100 pies

- How did you figure it out?
- Is there another way you could do it?
- How many cups of raspberries do I need to make 300 pies?
  - How did you figure it out?

D. Content: Equations (Representation)

Let's switch to another problem with apples and bananas. There are 3 times as many apples (a) as bananas (b). Is it true that $3a = b$ represents the apple-and-banana relationship? Why or why not?

- Let's go back to the situation with pies and raspberries. Should we say $p = 3r$ or $3p = r$?

$$3p = r \qquad 3p = 63$$

- How are these equations similar? How are they different?

E. Content: Graphing (Meaning)

Without drawing an actual graph, what would the shape of the graph look like if I plotted the points in the table? Why?

- Can I connect the points on the graph? What would it mean?

If you were to graph the number of pies on the horizontal axis and number of cups of raspberries on the vertical axis, how would you be able to tell if the point (2, 6) was on the graph?

Which way would you prefer to use to find the number of pies that we could make if we had 144 cups of raspberries? Scaling in tandem? A graph? Or an equation?

E. Extension: Unit Rate

Consider *only* the numbers as given in the table of the original problem. What would 36 – 24 mean?

- What would $\frac{36-24}{12-8}$ mean?

CONVERSATION PATHS (ANNOTATED)*

A. Problem Solving: Orientation

How do you want to make sense of this problem?

The purpose of this question is to get a sense of what students understand about the problem upon an initial reading of it. For example, a student may say "I'm going to look for a pattern in the table."

- How many quantities are there in the question? What are they?

  There are two quantities: the number of pies and the number of cups of raspberries needed for that number of pies.

- Do the quantities seem to vary with each other? If so how?

  Yes. If the number of pies changes by a factor of k, the number of cups of raspberries changes by the same factor of k. Similarly, the number of cups of raspberries is always 3 times the number of pies.

B. Problem Solving: Strategy

What patterns do you see in the measurement data? What is the relationship between cups of raspberries and pies?

For example, the number of pies is always ⅓ the number of cups of raspberries, or the number of cups of raspberries is always 3 times the number of pies.

- What do each of those patterns mean?

  The quantities "scale in tandem" so they have a ratio relationship.

- How is knowing this helpful?

  Given this relationship, we can determine any number of cups of raspberries needed to make any number of pies.

- Could we use this relationship between the number of pies and the number of cups of raspberries to find any pair of measures for the pies and raspberries?

  Yes. One could, for example, multiply the number of pies by 3 to find the number of cups of raspberries.

- Explain how the measures for 8 pies and 20 pies are related.

  20 pies are 2.5 times as many pies as 8, and 20 pies uses 2.5 times as many cups of raspberries as 8 pies—60 cups versus 24 cups.

C. Content: Proportional Relationship

How many pies can I make with 300 cups of raspberries?

100 pies

- How did you figure it out?

  Since 1 pie requires 3 cups of raspberries, 100 times as many pies will require 300 times as many cups of raspberries.

- Is there another way you could do it?

  Since the number of pies is ⅓ times as large as the number of cups of raspberries, 300 cups of raspberries will make 100 pies.

- How many cups of raspberries do I need to make 300 pies?

  900 cups

  - How did you figure it out?

    1 pie requires 3 cups of raspberries, so 100 times as many pies will require 300 times as many cups of raspberries.

D. Content: Equations (Representation)

Let's switch to another problem with apples and bananas. There are three times as many apples (a) as bananas (b). Is it true that $3a = b$ represents the apple-and-banana relationship? Why or why not?

It does not represent the relationship. For example, if $a = 10$, $b = 3 \times 10 = 30$. This shows that there are 3 times as many bananas as apples.

- Let's go back to the situation with pies and raspberries. Should we say $p = 3r$ or $3p = r$?

$3p = r$, since we need to multiply the number of pies by 3 to obtain the corresponding number of cups of raspberries needed to make that many pies.

- How are these equations similar? How are they different?

$$3p = r \qquad 3p = 63$$

In each case, the value on the right side represents 3 times the value of p. On the left, there are many solutions (infinite, actually), and larger values of one variable will correspond to larger values of the other. The unknowns can vary. On the right, there is only one value for p that can make the equation true. It does not vary, like the unknowns in the left equation. We could see the equation on the right as a specific case of the equation on the left.

E. Content: Graphing (Meaning)

Without drawing an actual graph, what would the shape of the graph look like if I plotted the points in the table? Why?

Every time the number of cups of raspberries is increased by 3, we increase the number of pies by 1. Or, similarly, every time we increase the number of pies by 1, we increase the needed cups of raspberries by 3. The points would fall in a line.

- Can I connect the points on the graph? What would it mean?

We could connect the points, but it would mean that all points on that line represent possibilities. So, 1.5 pies would go with 4.5 cups of raspberries since it would be on the line. That wouldn't really make sense, though.

If you were to graph the number of pies on the horizontal axis and number of cups of raspberries on the vertical axis, how would you be able to tell if the point (2, 6) was on the graph?

The 2 would have to represent the number of pies, and the 6 would represent cups of raspberries. Since that pair of measures can come from scaling in tandem or by noting that the pair of measures maintains the relationship of 3 cups of raspberries per pie, that point will be on the graph.

Which way would you prefer to use to find the number of pies that we could make if we had 144 cups of raspberries? Scaling in tandem? A graph? Or an equation?

The purpose of this question is to listen to students' understanding of relating two numbers, equations, tables, and graphs. Ask why they prefer one method over the others and what they understand about their preferred method.

E. Extension: Unit Rate

Consider *only* the numbers as given in the table of the original problem. What would 36 – 24 mean?

It would represent the increase in the number of cups of raspberries (12) when the number of pies increased by 4 (from 8 to 12).

- What would $\frac{36-24}{12-8}$ mean?

It compares the increase of 12 cups of raspberries to a corresponding increase of 4 pies. As a fraction, it shows the unit rate of 3 cups of raspberries per pie.

6.5 Shareables*

Use the information to complete the task.

The table shows the number of cups of raspberries, r, needed to make p pies using a baker's recipe.

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6 cups of raspberries are needed for 2 pies.	True	False
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The equation $p = 3r$ represents the relationship between r and p.	True	False
The point $(2, 6)$ would be on the graph that shows the relationship between the number of pies (shown on the horizontal axis) and the number of cups of raspberries (shown on the vertical axis).	True	False

D.

$$3p = r \qquad 3p = 63$$

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